Variable annuities: an example of derivatives use in life insurance

Frido Rolloos

Introduction

- The insurance market
  - Non-life insurance (also known as GI or P&C)
    - Health insurance
    - Home insurance
    - Accident & liability insurance
    - Marine & aviation, etc.
  - Life insurance
    - Death benefits
    - Living benefits
  - Re-insurance: insurance for (re-)insurance companies
- Insurance companies: AXA, Generali, SwissRe, and many more
- In this presentation
  - Embedded optionality in Variable Annuity products
  - The embedded options are typically:
    - Guaranteed Minimum Death Benefit (GMDB)
    - Guaranteed Minimum Accumulation Benefit (GMAB)
    - Guaranteed Minimum Withdrawal Benefit (GMWB)
    - Combinations thereof
  - Description, pricing and risk management of GMxBs
    - GMxBs can be extremely complex, we shall discuss only the most basic examples

GMDB product description I

- At inception (start) of the contract, the policyholder gives the insurance company a certain amount $A(t_0)$ (the ”premium”)
- This premium is invested in a basket of funds comprising of a mixture of fixed income and equity mutual funds: $A(t_0) = \sum_i n_i(t_0) F_i(t_0)$
- The evolution of this investment is called the “account value” $A(t_i)$
- Between inception and maturity date, policyholders could ‘lapse’ or die. When a policyholder lapses (also known as surrender) they forego any guarantees and receive the account value at surrender date less penalty fees (surrender charges)
- If the lapse rate in the interval $[t_i, t_{i+1}]$ is $\ell(t_i)$ and the mortality rate in the same interval is $m(t_i)$, then the ‘inforce’ at time t_k is

$$s(t_k) = \prod_{i=0}^{k-1} (1 - \ell(t_i))(1 - m(t_i))$$

- If the policyholder survives until contract maturity date T, the policyholder receives $A(T)$
- In the event of death of the policyholder before the maturity date T of the contract, the beneficiary receives a death benefit equal to $max(A(\tau), A(t_0)) = A_\tau + (A(t_0) - A(\tau))_+$, where τ is the time of death

GMDB product description II

- Instead of assuming a distribution for τ, in practice a mortality table is often used with a deterministic mortality rate $m(t)$ and also deterministic lapse rate $\ell(t)$. The price of the GMDB is then

$$GMDB = \sum_{i=0}^{N-1} s(t_i) m(t_i) E\left[e^{-\int_{t_0}^{t_{i+1}} r(u)du} (A(t_0) - A(t_{i+1}))_+ \middle| \mathcal{F}_{t_0} \right] \quad (t_N = T)$$

  where the expectation is under the risk-neutral measure
- The cost of the GMDB, in addition to management and other fees are deducted from the account value. If q is the fee rate per annum, then

$$A(t_{i+1}) = (1-q)A(t) + \sum_{j=1} n_j(t)(F_j(t_{i+1}) - F_j(t_i))$$

- Notice that the magnitude of the fees will influence the cost of the death benefit (just as dividends influence the price of put options). However, the fees include the cost of the GMDB. So this is an implicit pricing problem that can only be solved recursively using Monte Carlo simulations, even under Black-Scholes assumptions
- In many GMxBs the policyholder can rebalance the allocation to the different funds, within certain bandwidths, throughout the duration of the contract
- Why would a policyholder surrender?
  - A (sudden) rise in interest rates, so other insurance companies may be offering GMDBs at more attractive rates
  - Markets have rallied so much and the policyholder prefers to cash-in
  - The policyholder needs cash for other purposes

GMAB product description

- GMDBs are often sold together with GMABs
- A GMAB works similar to a GMDB, except that the payout occurs at maturity date T if the policyholder is alive at T
- The payoff of a GMAB is therefore

$$GMAB = s(T)E\left[e^{-\int_0^T r(u)du}(A(T) - A(t_0))_+ \middle| \mathcal{F}_{t_0} \right]$$

  where $s(T)$ is the percentage of the population that has not lapsed or died at T
- The above payoff is a simplification: $s(t)$ assumed to be deterministic
- For illustration the strike of the options has been taken to be $A(0)$. In practice the strike is often a floating strike, eg rolling-up at some agreed rate, ratchet features, lookback features
- A GMABs in combination with a GMDB is in fact a savings and potentially wealth accumulation product with a life insurance component

GM(L)WB product description

- As in a GMAB/GMDB, the policyholder gives the insurance company $A(0)$ which is invested in a basket of funds
- In a GMWB the policyholder receives coupons, say annually, throughout the tenor of the GMWB. For example, $A(0) = 100$ and over the next ten years, the duration of the GMWB, the policyholder receives 10 annually.
- The coupons are usually fixed amounts, not a percentage of the account value $A(t)$. Consequentially the account value can become 0 before maturity date, but insurance company still needs to pay coupons to the policyholder. This is the main risk of a GMWB
- In a GMLWB the maturity date is the death of the policyholder. In other words, coupon payments continue until the policyholder dies
- GMLWBs are therefore very much exposed to longevity risk
- In contrast to a GMAB, a GMWB is a retirement income product. A savings and retirement income structure could therefore be a GMAB + GMDB for ages 40 - 65 which is automatically converted to a GM(L)WB thereafter
- Typical cost of GMAB + GMDB or GMWB is around 150bps of account value $A(t)$ per annum
- Difficult to write analytical expression for payoff of GMWB, let's look at a basic pricing script

Code for a toy GMWB I

```
function ret = ToyWB()

vol = .15;                      % flat vol assumption
rf = .025;                      % flat interest rate assumption

m = .01;                        % mortality rate
l = .02;                        % lapse rate

T = 10;                         % contract duration

nsims = 1000;
nsteps = T;                     % annual time steps

iid = randn(nsims,nsteps);      % generate random variables

fee = 0.02;                     % total fee rate for policyholder, deducted from AV
cpn = 0.1;                      % GMWB annual coupon as a percentage of initial premium AV(0)
KDB = 100;                      % strike of GMDB

GMWB = [];
GMDB = [];
PVAV = [];
PVFees = [];

for i=1:1:nsims

 inforce = 1;
 AV0 = 100;
 AV = AV0;

 tmpWB = 0;
 tmpPVFees = 0;
 tmpDB = 0;
```

Code for a toy GMWB II

```
for j=1:1:nsteps

 AV = AV*exp(rf - 0.5*vol^2 + vol*iid(i,j));

 tmpPVFees = tmpPVFees + AV*(1-exp(-fee))*exp(-rf*j);

 AV = AV*exp(-fee);

 death = inforce*m;
 inforce = inforce*(1-m)*(1-l);

 tmpDB = tmpDB + max(KDB-AV,0)*death*exp(-rf*j);

 if AV >= cpn*AV0
   tmpWB = 0;
 elseif 0 < AV < cpn*AV0
   tmpWB = tmpWB + (cpn*AV0 - AV)*inforce*exp(-rf*j);
 else
   tmpWB = tmpWB + cpn*AV0*inforce*exp(-rf*j)
 end

 AV = max(AV - cpn*AV0,0);

end

GMDB = [GMDB;tmpDB];
GMWB = [GMWB;tmpWB];
PVFees = [PVFees;tmpPVFees];

end

GMWB = mean(GMWB);
GMDB = mean(GMDB);
PVFees = mean(PVFees);
```

GMxB main risks

- Actuarial risks
  - Mortality risk (GMDB)
  - Longevity risk (GMAB, GMWB)
  - Lapse risk (GMxB)
- Market risks
  - Basis risk (tracking error risk) from ‘fund mapping’:
    - GMxBs are derivatives on a basket of funds. No fund derivatives, so mapping fund returns to returns of tradable indices is required, but this mapping will not be perfect
  - Interest rate risk from discount curve and bond funds / FI indices exposure
  - Foreign exchange risk
  - Price risk
  - Volatility risk: what volatities do we mean?
  - Higher order risk: cross-gammas, vanna, volga
- Model risk
- Operational risk
  - Thousands of policies / insurance contracts to value
  - Data quality and availability
  - Infrastructure and staffing

GMxB market risk management

- GMxBs are long duration ($>$ 10yrs) path-dependent options on baskets of funds
- Risk management is complex and ultimately a cost-benefit exercise. More an art than science
- Market risk management begins with 'fund mapping' and a covariance matrix. The simplest fund mapping is a simple linear regression:

$$\frac{\Delta F_i}{F_i} = \sum_j \beta_{ij} \frac{\Delta(X_j I_j)}{X_j I_j} + \varepsilon_i$$

  where X_j are foreign exchange rates and I_j are tradable indices, preferably with an options market

- Fund mapping is a tool for:
  - Fund selection to reduce basis risk and ongoing monitoring of basis risk
  - Executing MC sims to value the option(s) and calculate option sensitivities
- In general the aim is to minimize the variability of ε_i and β_{ij} under the constraint that $\beta_{ij} \geq 0$ and $\sum_j \beta_{ij} = 1$. Roughly speaking this means
  - Funds with consistent 'style'
  - Indices should not be highly correlated
  - No leveraged funds (long only funds)

Hedge instruments

- Hedge strategies
  - Fully static hedge
    - Buy basket put options
    - Reinsure market risk
  - Semi-static hedging: hedge convexity with options, delta hedge linear risk
  - Delta-vega-rho linear hedging: this only covers linear risks
  - Cross-gammas frequently left open under semi-static and dynamic hedging
- Static hedging is expensive: another party carries the risk and will charge for it. In practice semi-static hedging or delta-vega-rho hedging
- Delta hedge:
  - Rolling position in equity Total Return Swaps and/or Forwards and FX forwards (OTC trades)
  - Daily hedging with equity index futures (Listed trades)
  - Under large market moves delta-hedging insufficient, additional vanilla options (listed or OTC) to hedge gamma / price risk convexity
- Rho hedging:
  - Two sources of IR risk: discount curve and bond fund indices
  - Interest Rate Swaps are preferred hedge instruments
  - Under large market moves linear rho hedging insufficient, additionally swaptions may be included to hedge IR convexity
- Vega hedging
  - In general variance swaps, volatility swaps and/or vanilla options used for vega hedging

In model vs out (of) model hedging

- Models, for example the Black-Scholes-Merton model, comprises of variables and parameters
- Variables evolve stochastically, whereas parameters are deterministic or constant
- In the BSM model

$$dI(t) = r(t)I(t)dt + \sigma(t)I(t)dW(t), \;\; dr(t) = a(r(t))dt, \;\; d\sigma(t) = b(\sigma(t))dt$$

  the variable is the index price $I(t)$ and the parameters are $r(t), \sigma(t)$. In the Heston model

$$dI(t) = r(t)I(t)dt + \sigma(t)I(t)dW(t), \;\; dr(t) = a(r(t))dt, \;\; d\sigma^2(t) = \kappa(\theta - \sigma^2(t))dt + \eta\sigma(t)dZ(t)$$

  $I(t), \sigma(t)$ are variables, all others are parameters
- In model hedging: Hedge according to model, it other words parameters are not hedged even though they may vary (eg calibration)
- Out of model hedging: Hedge both the stochastic variables as well as (a subset of) parameters. Eg under BSM hedging IR and vola risk is out of model hedging
- Pros and cons: Out of model hedging can simplify the hedge strategy and reduce computational time. The problem is that there will be a mismatch between in model price and out of model hedge/replication cost

Vega hedging

- Vega hedging as an example of in model and out of model hedging
- What do we mean by ‘vega’? Sensitivity with respect to
  - Implied volatility?
    - What is the strike of a GMWB? What is its implied volatility?
    - Even if we can define IV for a GMWB, how does it relate to IV of vanilla options (hedge instruments)?
  - Instantaneous volatility?
    - Arguably the correct theoretical way to define vega
    - However instantaneous volatility is not really an observable, so is it robust?
    - Furthermore, vega defined this way is model dependent
  - Variance swap strike?
    - Varswaps are observables and in fact almost model-free
    - How do you calculate sensitivity of a GMxB wrt varswap strike?
    - Varswaps are expensive, and not liquidity depends on the index
  - Volatility swap strike?
    - Volswaps are not model free
    - Also difficult to calculate sensitivity wrt volswap
    - But less expensive than varswaps
- How we define vega affects the choice of hedge instrument
- There are no easy answers. Hedging is sometimes more art than science

Q&A