QUANTITATIVE FINANCE MADE ACCESSIBLE

COURSE N°5:

« Interest Rate Modeling»

Saturday, October 28, 2023

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FINANCE TUTORING

Conseil et Formation

AGENDA FOR THE SESSION

- **Course n°5:** ***« Option pricing models »***
- ***Welcome to all participants of course number 5 in the webinar series titled "Quantitative Finance Made Accessible." Today's topic is Interest rate models***
- Understanding how interest rates change is crucial for financial institutions, governments, and individual investors. The study of interest rate modeling allows us to capture the behavior of interest rates and helps in pricing various financial instruments. In this course, we will delve deep into some of the most widely used models for interest rate forecasting.

AGENDA FOR THE SESSION

- **Course n°5: *« Option pricing models »***

- I’m pleased to introduce our guest speaker, Summary of Guest Speaker Experience:
- Our guest speaker brought a unique blend of theoretical and practical insights into the world of finance, particularly around trading credit risk models used in banks. Hailing from a renowned institution, UBS, the speaker currently functions as a Quant, specializing in XVA models. These models play a pivotal role in determining the Pillar-1 capital for the bank.
- 15-minute talks followed by a 15-minute question and answer session

I. SHORT-RATE MODELS

THE VASICEK MODEL

INTRODUCTION

- The Vasicek Model was one of the first attempts to describe the behavior of interest rates using a stochastic differential equation. Its introduction was groundbreaking because, before this, the majority of models relied heavily on deterministic methods.

THE FEATURES OF THE VASICEK MODEL

❖ **The features of the vasicek model:**

- **Mean Reversion:** One of the model's core assumptions is that interest rates tend to revert to a long-term mean over time. This means that if the interest rate is above its long-term mean, it is expected to decrease, and vice versa.
- **Randomness:** The model incorporates a random factor, representing the unpredictable nature of interest rates due to various external factors.
- **Parameters:** The Vasicek Model is defined by three parameters:
  1. Long-term mean interest rate.
  2. Speed at which the interest rate reverts to the mean.
  3. Volatility, which captures the uncertainty or the extent to which the interest rate can fluctuate.

THE MATHEMATICAL MODEL

❖ The mathematical model

- Mathematically, the model can be described by the stochastic differential equation:

$$dr_t = a(b - r_t)dt + \sigma dW_t$$

Where:

- r_t is the short rate at time t
- dr_t is the change in the interest rate.
- a is the speed of mean reversion.
- b is the long-term mean interest rate.
- σ is the volatility.
- dW_t is the Wiener process, representing the random shock.
- $a(b - r_t)dt$: **deterministic term**
- σdW_t: **diffusion term**

THE MATHEMATICAL MODEL

❖ **The mathematical model**

- The term $a(b - rt)$ represents the force that pulls the short rate towards the long-term mean.
- When the long-term rate b is higher than the short-term rate *rt*, $b - rt$ is positive.
  - This makes $a(b - rt)$ positive, meaning the drift component of the short rate is positive.
  - As a result, the expected change in the short rate *drt* will be positive.
- This means the model expects the short rate to increase over time.

And conversely….

EXAMPLE

❖ **The mathematical model**

- Let's walk through a basic numerical example of how the Vasicek Model might predict short-term interest rates.
- **Given Parameters:**

✓ Long-term mean interest rate, $b = 5\%$ or 0.05

✓ Speed of mean reversion, $a = 0.3$

✓ Volatility, σ$\sigma = 2\%$ or 0.02

✓ Initial short rate, $r0 = 4\%$ or 0.04

3. EXAMPLE

- Assume we want to forecast the interest rate one year from now and during that year there is a random shock represented by $dWt = 0.01$.

- Using the Vasicek formula:

$$drt = a(b - rt)dt + \sigma dWt$$

- Inserting our values for a small time increment (let's use $dt = 1$ year for simplicity):

$$\boldsymbol{drt = 0.3(0.05 - 0.04) + 0.02(0.01) = 0.0032}$$

- Now, add this change to the initial rate to get the forecasted rate after one year:
- $r1 = r0 + drt = 0.04 + 0.0032 \; = 0.0432$

So, the predicted interest rate one year from now, given the random shock and parameters provided, is 4.32%.

3. EXAMPLE

- ❖It's important to note that in real-world applications, interest rate paths would be simulated many times to capture the stochastic nature of the Vasicek Model. The example above is a simplified version to provide a basic understanding of how the model functions numerically.

I. SHORT-RATE MODELS

THE COX-INGERSOLL-ROSS (CIR) MODEL

INTRODUCTION

- The CIR model sought to rectify some of the limitations of the Vasicek model, notably preventing negative interest rate values which can be theoretically produced in the Vasicek model. The model was developed in the context of an increasing need for more realistic and accurate ways to price bond options and other interest rate derivatives.

THE FEATURES OF THE CIR MODEL

❖**Features of the CIR model:**

- **Positivity:** One of the main features of the CIR model is that it incorporates a square root term, which ensures that the short rate remains non-negative.
- **Mean Reversion:** Like the Vasicek model, the CIR model also has mean-reverting properties. However, the speed and manner of mean reversion can be different due to the square root term.
- **Stochastic Volatility:** The volatility term in the CIR model is stochastic, varying with the level of the interest rate, making it more adaptable to the actual behavior of interest rates.Long-term mean interest rate.

2. THE MATHEMATICAL MODEL

2. The mathematical model

- The CIR model can be expressed as the following stochastic differential equation:

$$\boldsymbol{drt = a(b - rt)dt + \sigma\sqrt{rt}dW}$$

➢ Where:

- $\boldsymbol{rt}$ is the short rate at time t
- $\boldsymbol{drt}$ is the change in the interest rate..
- $\boldsymbol{a}$ is the speed of mean reversion.
- $\boldsymbol{b}$ is the long-term mean interest rate.
- $\boldsymbol{\sigma}$ is the volatility.
- $\boldsymbol{dWt}$ is the Wiener process, representing the random shock.
- $a(b - rt)dt$: **deterministic term**
- $\sigma\sqrt{rt}dW$: **diffusion term**

3. CONCLUSION

- The CIR model, with its non-negativity feature, offers a more realistic framework for modeling interest rates, especially in low-rate environments. It's widely used in the finance industry, especially for pricing interest rate derivatives and for risk management purposes.

I. SHORT-RATE MODELS

THE DETERMINATION OF:

a, is the speed of mean reversion
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a, Is the speed of mean reversion
b, the long-term mean interest rate
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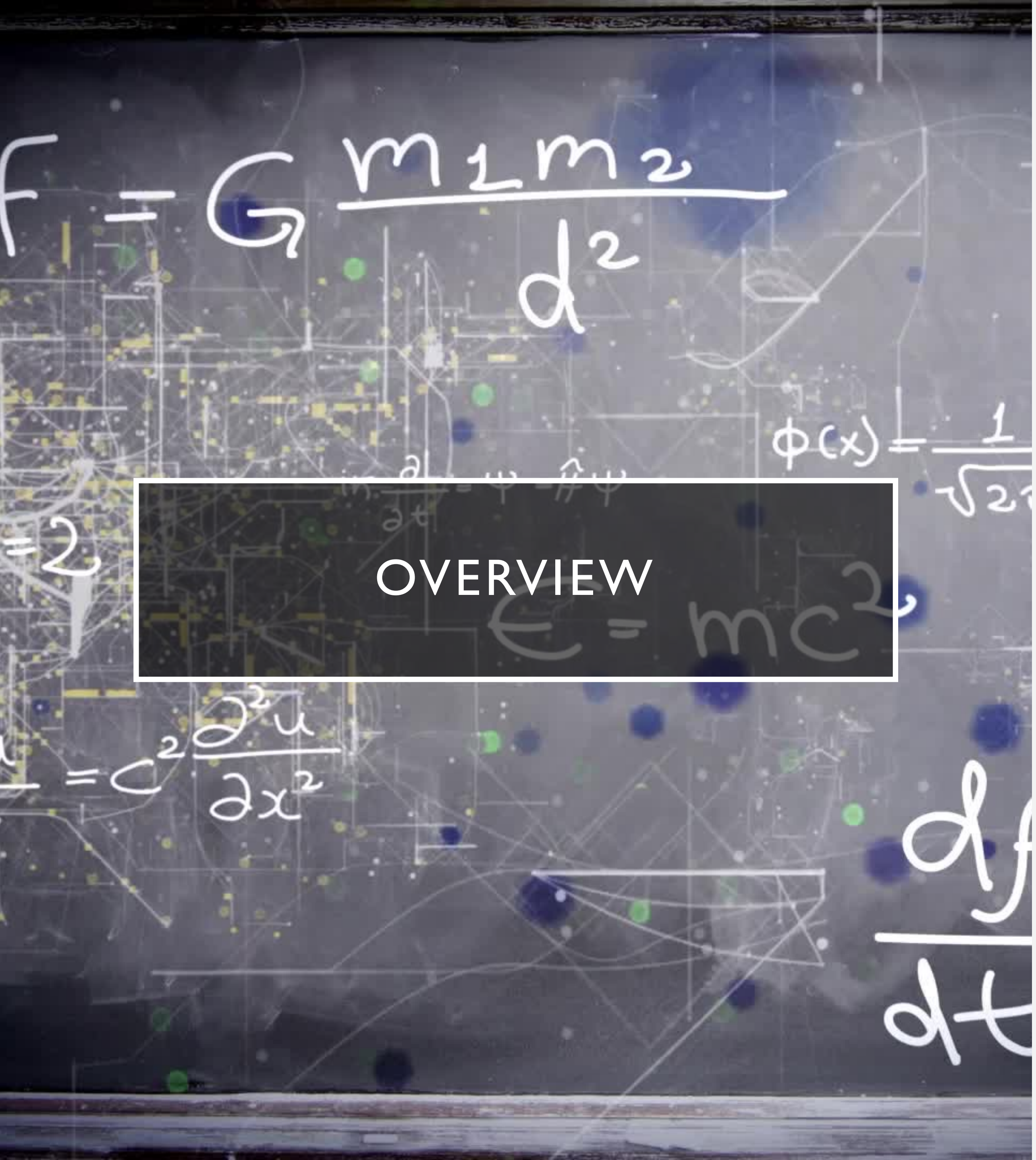
- The parameters a, b, and σ in the Cox-Ingersoll-Ross (CIR) model are determined empirically, based on fitting the model to observed data. Here's a general overview of how they might be determined:

1. **Historical Data**: Gather historical interest rate data or the data of the financial quantity you are trying to model. This could be daily, weekly, or monthly data, depending on the application.
2. **Objective Function**: Define an objective function that measures the difference between the observed data and the values predicted by the CIR model. This could be the sum of squared differences or some other measure of fit.
3. **Optimization**: Use an optimization algorithm to find the values of aa, bb, and σσ that minimize the objective function. This can be done using techniques such as Maximum Likelihood Estimation (MLE) or Nonlinear Least Squares (NLS).
4. **Model Validation**: Once you have estimated parameters, it's important to validate the model using out-of-sample data or other statistical tests to ensure its accuracy and appropriateness.

II. TERM STRUCTURE MODELS

SHORT RATE MODELS VS. TERM STRUCTURE MODELS



OVERVIEW

❖ **Short Rate Models vs. Term Structure Models:**

- <u>Short Rate Models:</u>

✓ Focus on instantaneous interest rate, $r(t)$.

❑ **Examples**: Vasicek, Cox-Ingersoll-Ross

✓ Used for pricing derivatives sensitive to the short end of the yield curve.

✓ Typically single-factor models.

✓ Simpler and more analytically tractable.

- <u>Term Structure Models:</u>

✓ - Describe the entire yield curve for different maturities.

❑ **Examples**: Heath-Jarrow-Morton, LIBOR Market Model.

✓ Used for instruments depending on multiple points on the yield curve.

✓ Often multi-factor models capturing several sources of randomness.

✓ More complex and might require advanced numerical methods.

III. TERM STRUCTURE MODELS

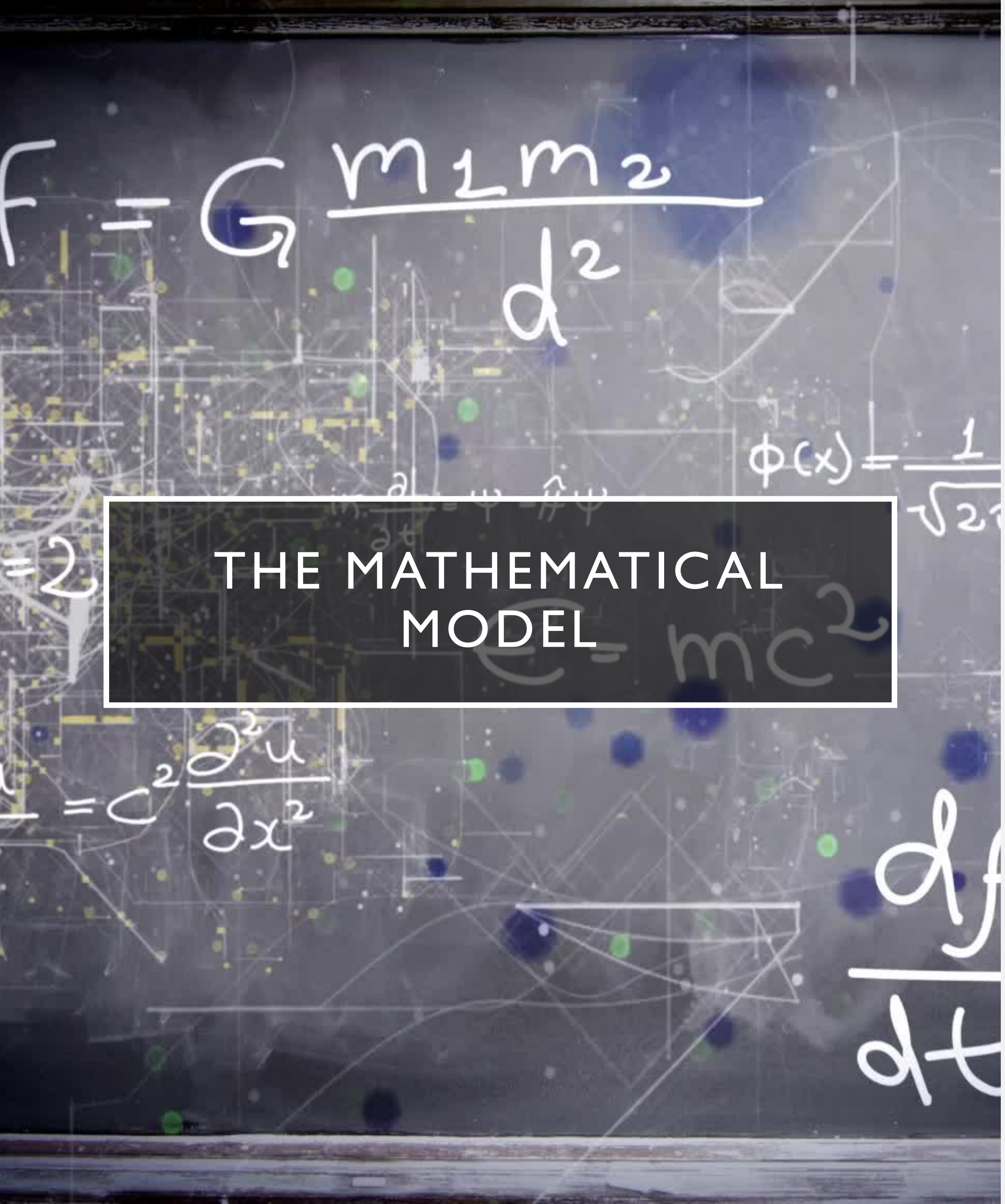
HEATH-JARROW-MORTON (HJM) FRAMEWORK

HEATH-JARROW-MORTON (HJM) FRAMEWORK

❖ **Heath-Jarrow-Morton (HJM) Framework:**

•Offers a general framework to model the evolution of the interest rate term structure (or yield curve).

•Stands out from other models as it directly models forward rates instead of the instantaneous short rate.



❖ **The mathematical model**

➢ Given a forward rate $f(t,T)$ for some fixed time T and any $t < T$, its evolution under HJM is given by:

$$\boldsymbol{df(t,T) = \mu(t,T)dt + \sigma(t,T)dW(t)}$$

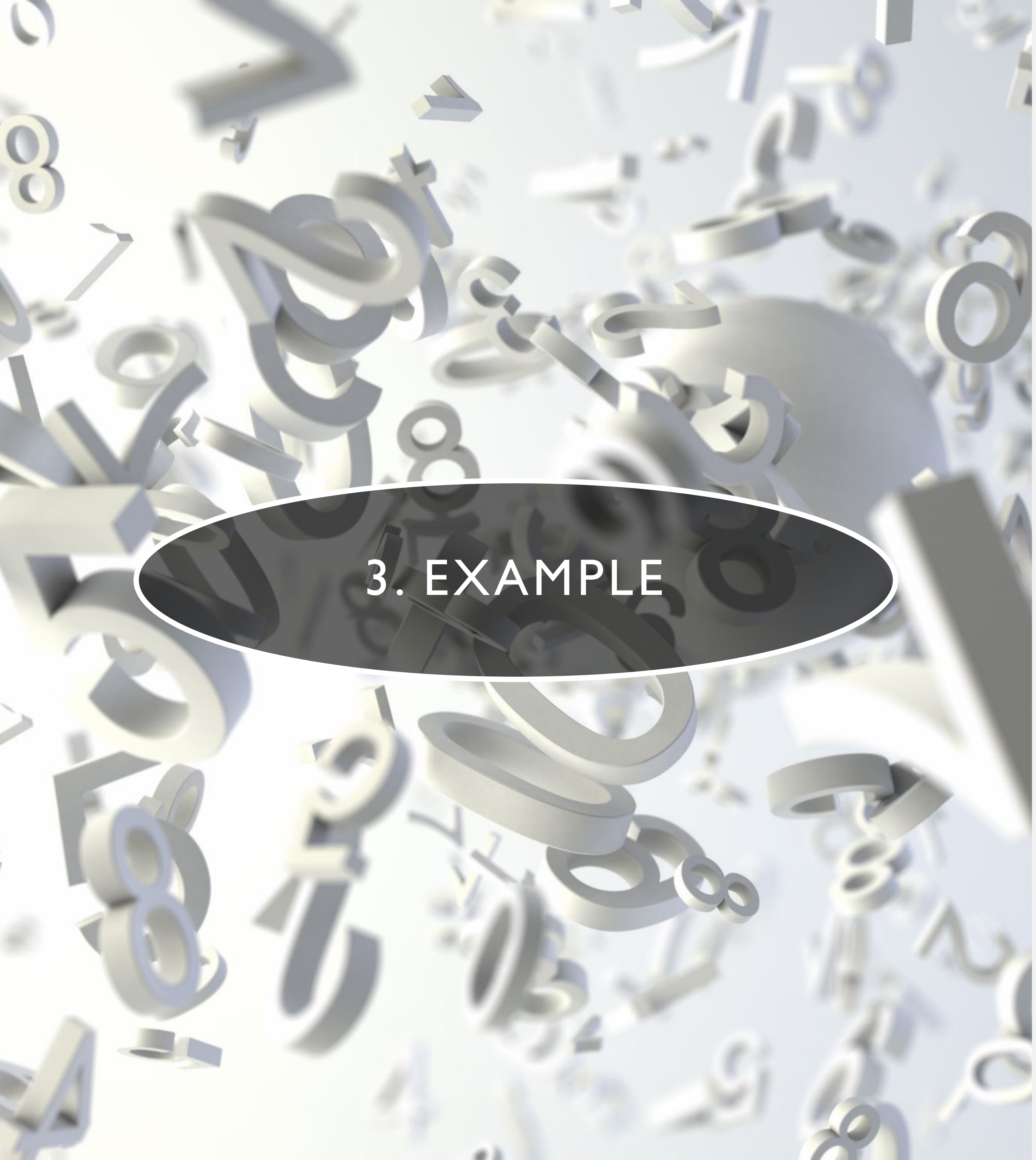
Where:

•$\mu(t,T)$ is the deterministic drift term ensuring absence of arbitrage.
•$\mu(t,T) = \sigma(t,T)\int \sigma(t,u)du$

•$\sigma(t,T)$ is the volatility of the forward rate.

•$dW(t)$ is a standard Brownian motion.

- The no-arbitrage condition leads to a drift term given by:



The mathematical model

- Suppose we have a forward rate curve for two future time points. We'll look at one-year forward rates:
- $f(0,1) = 5\%$: Forward rate starting one year from now, ending two years from now.
- $f(0,2) = 6\%$: Forward rate starting two years from now, ending three years from now.
- Given the HJM framework, the drift of the forward rate is determined by the volatility of the forward rates. Let's assume a simple deterministic volatility structure:
- $\sigma(1) = 1\%$ for f(0,1)$f(0,1)$
- σ(2)$\sigma(2) = 1.5\%$ for f(0,2)$f(0,2)$

3. EXAMPLE

- ❖ Assume we want to forecast the interest rate one year from now and during that year there is a random shock represented by $dWt = 0.01$.

- ➢ Using the Vasicek formula:

$$drt = a(b - rt)dt + \sigma dWt$$

- Inserting our values for a small time increment (let's use $dt = 1$ year for simplicity):

$$\boldsymbol{drt = 0.3(0.05 - 0.04) + 0.02(0.01) = 0.0032}$$

- Now, add this change to the initial rate to get the forecasted rate after one year:
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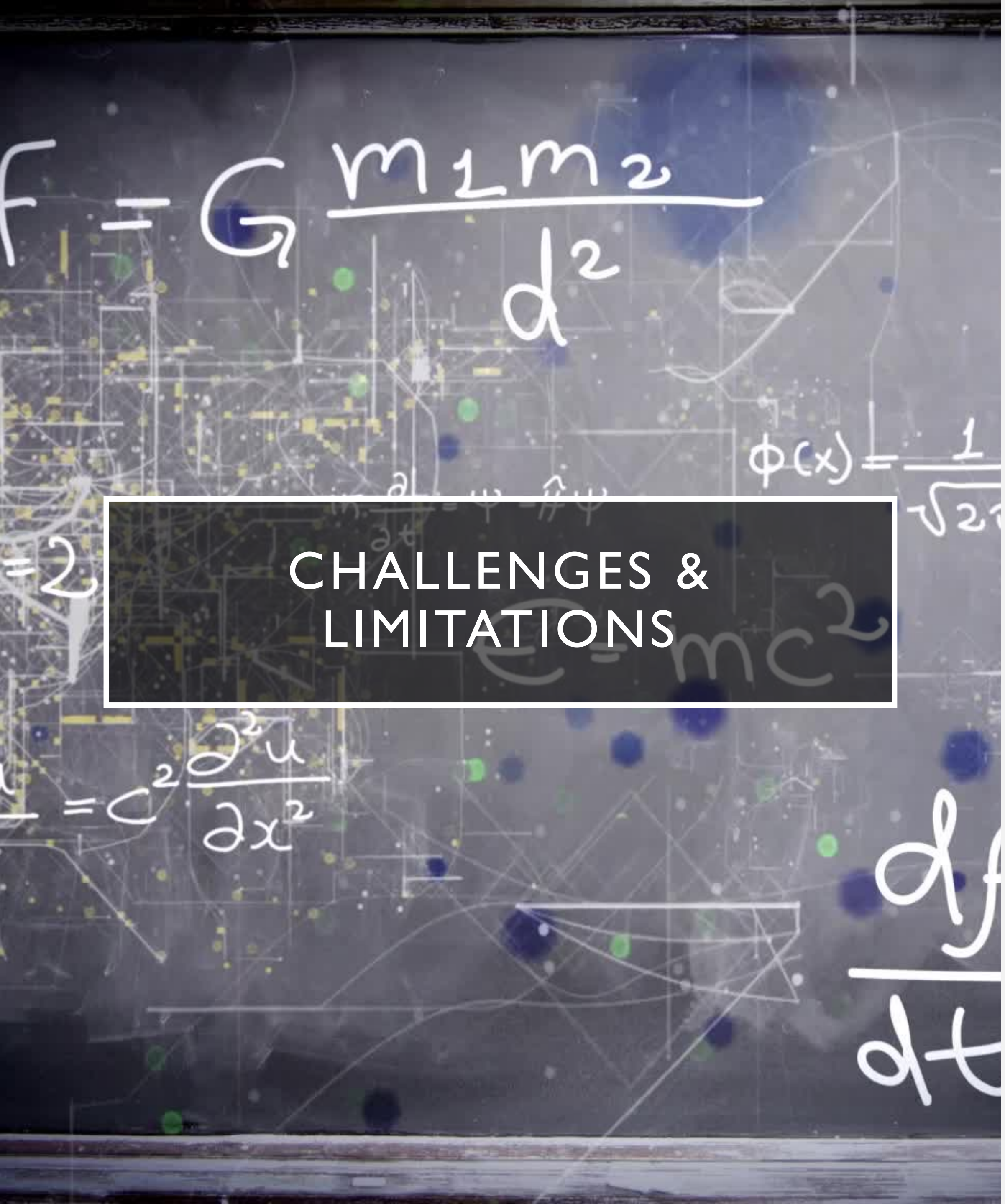
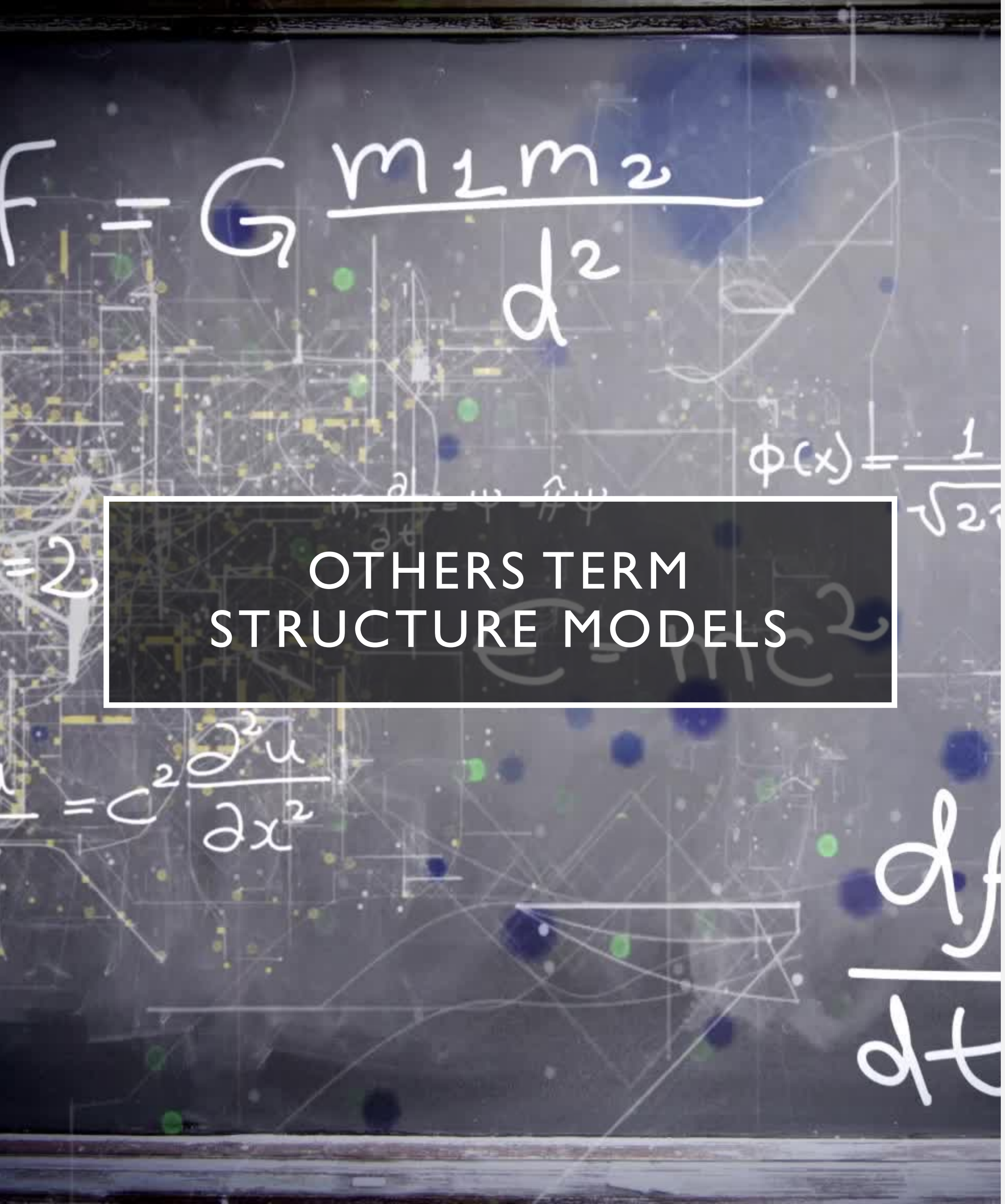


❖ Challenges & Limitations

•The HJM Framework is a comprehensive tool for interest rate modeling, known for its versatility, no-arbitrage feature, adaptability to market data, and ability to incorporate multiple risk factors using various Brownian motions.

•However, it's more intricate than traditional short-rate models and demands sophisticated numerical methods. It's particularly employed in pricing interest rate derivatives, managing risk in vast fixed income portfolios, and handling financial institutions' assets and liabilities.

•Unlike models like Vasicek and CIR, which target the immediate short rate, HJM examines the full yield curve evolution, broadening its view on rate shifts at the expense of higher complexity.



OTHERS TERM STRUCTURE MODELS

❖ Challenges & Limitations

- **Libor Market Model (LMM)**: This model focuses on the dynamics of the set of forward LIBOR rates. It is especially popular for pricing and hedging interest rate derivatives, particularly interest rate swaps and swaptions.

- **Hull-White Model**: This is an extension of the Vasicek model, introducing a time-dependent mean reversion level. It's a one-factor model that can be calibrated to fit both the current term structure of interest rates and its volatility structure.

THANK YOU…